APPLICATION OF TWO-STAGE OPTIMIZATION DANTZING-WOLF DECOMPOSITION METHOD[4]

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In today` s world of globalization, a competition war will be "survived" only by the companies who are able to realize their production on the market. If a company wants to realize its production on the market, it has to offer goods of minimal price and maximum quality at the same time. A tool how to fulfill this is optimization.[5] In our case it will be the company` s profit maximization. A simple optimization, as known in operation analysis, will not be sufficient; the problem will have to be solved in a complex way, i.e. we will have to take into consideration some facts brought by globalization. First of all, it is the formation of huge multinational companies, e.g. Slovnaft – MOL, VSŽ – US Steel, SPP – Gazprom, etc. Then we will not optimize an individual company production process, but a multinational company as whole[6], i.e. we will apply one of more-stage decomposition optimization method algorithms. In literature there are two decomposition method algorithms described, the Dantzing-Wolf algorithm and the Kornai-Lipták algorithm. We will concentrate on the first one.

Key words: *Dantzing-Wolf Algorithm, decomposition method algorithm*

1. Dantzing – Wolf algorithm

1.1. General characteristics of a decomposition algorithm

One of the basic algorithms used in linear programming decomposition assignments is the Dantzing and Wolf algorithm. The algorithm got its name following the book by G. B. Danzing *Linear programming and its development* where he states that P. Wolf was the first one to find out that a decomposition algorithm can be seen as a special case of '*a generalized linear program*', and therefore the name of algorithm must be identified with his name as well.

Practical problems of planning and management usually contain a great number of variables. The solution components are interconnected with many conditions expressing for example the need to satisfy demand, the need to fulfill various balance relations, the sources and other production factors boundaries, etc. A single economy unit` s results are influenced by particular superior demands, determined by common limited sources or the behavior of a examined economy unit as a whole.

In complex economy systems it is rather complicated to record and process all detailed relations in one centre. Assignments and responsibility must be shared. So there is an important task of organizing activities. The process of coordination for any system structure is the process of integration by which activities are getting more and more precise. Decomposition algorithms rank among basic approaches in which optimal programming assignment solution of great extend is substituted by a consecutive solution of smaller extend tasks.

In a system it is necessary to collect and process information about subsystems work and their technology potential in one centre. Thus a complex and laborious assignment arises, because in collecting and processing information of great extend the questions of reliability and currency of some gained information come to focus. The organization of information transfer between the centre and the subsystems must be carried out according to the following principle: a central body evolves instructions (regulations) for economy objects following rough description of subsystems technology potential and on the basis of its criteria and conditions. Following these instructions the subsystem gives precision to its potential. The new information about the subsystem potential is taken into consideration by the parent company, which then coordinates the instructions. This information exchange takes place until a solution close to an optimal one (from the whole economy system criteria point of view) is acquired.

The algorithm is designed to solve the following linear programming problem:

Maximize

$$L(X) = \sum_{p=1}^{k} C_p^T X^p \tag{1}$$

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[5] a particular input or output variable maximization or minimization

[6] i.e. we will optimize the parent and the subsidiary company at the same time

Subject to

$$\sum_{p=1}^{k} A_p X_p = \mathrm{b}^0 \qquad (2)$$

$$\mathrm{BpXp = bp} \qquad p = 1, 2, ..., k \qquad (3)$$

$$\mathrm{Xp} \geq 0 \qquad p = 1, 2, ..., k \qquad (4)$$

Where: A_P - is the matrix type m_0 x n_p,
B_p - is the matrix type m_p x n_p,
C_p - is n_p – sized column vector,
b_p - is m_p – sized column vector,
X_p - is n_p – sized variable column vector

The basic idea of the Dantzig-Wolf decomposition method is based on the following reflections. Let` s give the symbol S_P (p = 1, 2, ..., K) to a set of acceptable linear programming task solution. This task is the p-stage sub problem to the tasks (1) to (4)

maximize

$$L_p(X_p) = C_p^T X_p \qquad (5)$$

subject to

$$B_p X_p = b_p, \qquad (6)$$

$$X_p \geq 0$$

Thus

$$S_p = \{X_p \,/\, B_p X_p = b_p; X_p \geq 0\}$$

The task (1) to (4) will be called X-stage task and the task (5) and (6) X_p-stage task. Let N_p be the number of all outer points of the set S_p. Let X_{pj} (j = 1,2,...,N_p) be all outer points of the S_p, which are supposed to be known. Every point X of the acceptable solutions set to the problem X can be written in the form X = (X_1, X_2 ,..., X_k), where every X_p (p= 1, 2,...,k) is an acceptable solution for the X_p task. Any X_p can be expressed as a convex combination of the S_p set outer points X_{pj} , i.e.;

$$X_p = \sum_{j=1}^{N_p} z_{pj} X_{pj}, \qquad p = 1,2,...,k \qquad (7)$$

where z_{pj} are the convex combination coefficients

$$\sum_{j=1}^{N_p} z_{pj} = 1, \qquad p = 1, 2,..., k \qquad (8)$$

$$z_{pj} \geq 0, \qquad p = 1,2,...,k, j = 1, 2, ..., N_p \qquad (9)$$

Let` s substitute X_p in the X task function (1) and subject to (2) with the right side formula of the relation (7). We will get a new form of the problem X with the unknown z_{pj} . The matrix of the problem is:

Maximize

$$L(X) = \sum_{p=1}^{k} C_p^T \sum_{j=1}^{N_p} z_{pj} X_{pj}$$

subject to

$$\sum_{p=1}^{k} A_p \sum_{j=1}^{N_p} z_{pj}\ \mathrm{X}_{pj} = \mathrm{b}_0$$

$$\sum_{j=1}^{N_p} z_{pj} = 1,\ \mathrm{p} = 1, 2, \dots, \mathrm{k}$$

$$z_{pj} \geq 0, \qquad p = 1,2,\dots,k,\ j = 1, 2, \dots, \mathrm{N}_p$$

after the modification we will get the task
Maximize

$$\mathrm{L(X)} = \sum_{p=1}^{k} \sum_{j=1}^{N_p} z_{pj}\ \mathrm{C}_p^T \mathrm{X}_{pj} \tag{10}$$

subject to

$$\sum_{p=1}^{k} \sum_{j=1}^{N_p} z_{pj}\ \mathrm{A}_p \mathrm{X}_{pj} = \mathrm{b}_0$$

$$\sum_{j=1}^{N_p} z_{pj} = 1,\ \mathrm{p} = 1, 2, \dots, \mathrm{k}$$

$$z_{pj} \geq 0, \qquad p = 1,2,\dots,k,\ j = 1, 2, \dots, \mathrm{N}_p$$

As we suppose that the vector C_p, matrixes A_p and vectors X_{pj} are known, let` s make the record simple by writing the product $\mathrm{C}_p^T \mathrm{X}_{pj}$ as a number d_{pj} and the product $\mathrm{A}_p \mathrm{X}_{pj}$ as a vector, i. e.

$$\mathrm{d}_{pj} = \mathrm{C}_p^T \mathrm{X}_{pj} \tag{11}$$

$$\mathrm{P}_{pj} = \mathrm{A}_p \mathrm{X}_{pj} \tag{12}$$

The problem (10) will get an equivalent form with the unknown z_{pj}
maximize

$$\mathrm{L(X)} = \sum_{p=1}^{k} \sum_{j=1}^{N_p} d_{pj}\ \mathrm{z}_{pj} \tag{13}$$

subject to

$$\sum_{p=1}^{k} \sum_{j=1}^{N_p} P_{pj}\ \mathrm{z}_{pj} = \mathrm{b}_0 \tag{14}$$

$$\sum_{j=1}^{N_p} z_{pj} = 1, \qquad \mathrm{p} = 1, 2, \dots, \mathrm{k} \tag{15}$$

$$z_{pj} \geq 0, \qquad p = 1,2,\dots,k,\ j = 1, 2, \dots, \mathrm{N}_p \tag{16}$$

This problem will be called Z task. The optimal solution for the Z task unequivocally determines the optimal solution for the X task. If z^*_{pj} are the optimal Z-task solution components, then vectors

$$\mathrm{X}^*_p = \sum_{j=1}^{N_p} z_{pj}\ \mathrm{X}_{pj} \tag{17}$$

form the optimal initial X task solution as follows:

$$\mathrm{X}^* = (\mathrm{X}^*_1, \mathrm{X}^*_2, \dots, \mathrm{X}^*_k) \tag{18}$$

2. 2. Dantzing-Wolf method algorithm

The principle of the Dantzig - Wolf, which allows the simulation of the process of specifying plans in a two-stage hierarchical economy system, is that the solving of this task is replaced by successive solving of more LPT of smaller extend. This method is characterized by using so called master problem (coordination problem, centre problem) and interactive information exchange between this problem and the set of mutually independent sub problems, which stand for individual subsystems until the optimization criteria is fulfilled.

Auxiliary stage

Step 1- determine a block - diagonal structure of the problem i. e. to define

$$c^p, A_p, B_p, b^0, b^p, x^p$$

Step 2 - find at least one outer point for each of the sets

$$S_p = \{x^p/B_p x^p \leq b^p, x^p \geq 0\} \quad p = 1,...,k$$

Step 3 - calculating

$$d_{pj} = c^p \; X_{pj}$$
$$P_{pj} = A_p \, X_{pj}$$

for all the known outer points

Step 4 - the master problem construction

$$L(z) = d_{11} z_{11} + d_{12} z_{12} + ...d_{pj} z_{pj} \to \max...$$
$$p_{11} z_{11} \qquad P_{pj} Z_{pj} = b^0$$
$$z_{11} \qquad = 1$$
$$z_{12} \qquad = 1$$
$$Z_{pj} = 1$$
$$Z_{pj} \geq 0$$

Main stage

Step 1 - Master problem OS calculation

primary $Z^x = (Z^x{}_{pj})$

dual $y = (u_1...u_m, v_{1...}v_2)$

u – dual OS prices vector for global sources

v – total p- enterprise profit evaluation on the level of company

Step 2 - Construction and problem solving in subsystems

$$p = 1,...,h$$
$$L_p(x^p) = (c^p - uA_p)\, x^p \to \max$$
$$B_p x^p;\; x^p \geq 0$$
$$\text{OS} \qquad L_p(x^p) = L^*_p,$$

Step 3 - $(v_r - L_r) = \min \{v_p - L^*_p,\; p = 1...h\}$

Step 4 - $(v_r - L_r) < 0$

$\to$ NO $\Rightarrow$ the construction of the initial problem OS

$$x = (x^1...x^4)\;,\; x^p = \sum_{p=1}^{k} X_{pj} Z_{pj} \qquad p = 1...h$$

$\to$ YES $\Rightarrow$ ***step 5***

Step 5 - Master problem extension with the variable Z_{rj} (column)

$$P_{rj} = \begin{pmatrix} P_{rj} \\ e_r \end{pmatrix} \qquad P_{rj} = A_r X_{rj}\; d_{rj} = c^r x_{rj}$$

e_r is k- sized unit vector with the unit in r -place

$\to$ ***Step 1***

2.3. Dantzig-Wolfov algorithm and its application

The importance of the Dantzing-Wolf lies in its calculation efficiency and the possibility to give a concise economy interpretation of the optimal plan for the production complex. This complex consists of several subsystems interconnected by the means of the boundary summary shared by the whole complex and by a unique development goal.

The Dantzing-Wolf algorithm can form the basis for a production system common planning organization schedule in which the plan for the whole system is being improved by the means of specifying individual subsystems plans based on aggregated information exchange between the centre and the subsystems.

Let's suppose that the problem is interpreted as the problem of searching for a production plan of the company consisting of several branches with the aim of guarantying the company the highest profit with the sources limits being exceeded.

In accordance with the examined decomposition method the mutual relations between the branched and the company are in terms of standard planning organized as follows. each of k branches delivers the company a production plan draft (minimum one option):

$$X_{pj},\ p = 1, ..., k,\ j = 1, 2, ...$$

On the basis of these options the company management creates so-called master problem (the centre problem) in which the global company sources limits are taken into account as well. its solution is the values

$$\mathbf{Z}_{pj},\ p = 1, ..., k,\ j = 1, 2, ...$$

which can be interpreted as a j-plan option for a p-branch intensity.. thus a plan for the p-branch is settled:

$$X^{p} = \sum_{j=1}^{2...} Z_{pj} . X_{pj} \quad , P = 1, ... , K$$

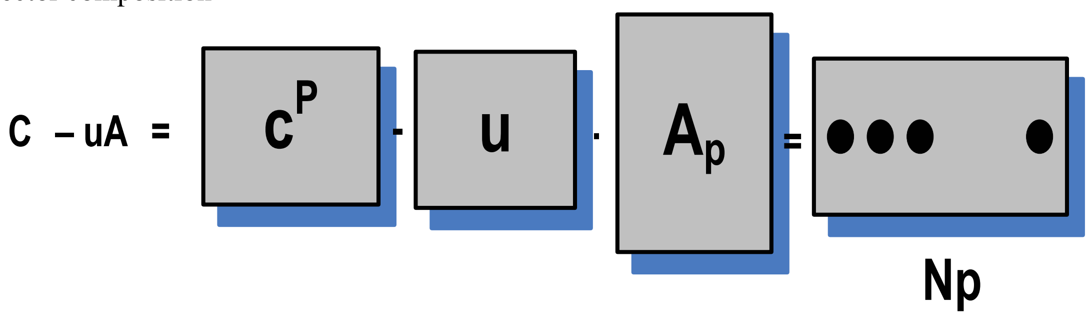
By solving the master problem the management obtains a dual task optimal solution as well

$$y = (u, v_1, ..., v_k) = (u_1, ..., u_m, v_1, ..., v_k),$$

while the vector u is interpreted as the optimal solution dual prices vector for global sources and the parameter v_p, p = 1, ..., k can be interpreted as the total p-branch profitability at the company level. These all company optimization results are communicated to the branches. The dual task constraints in respect to the master problem can be written in the following form:

$$v_{p} \geq (c^{p} - uA_{p}). X_{pj} \quad , \text{pre } \forall \ p, j$$

every ● vector composition



determines the profit, which we get producing one unit of a corresponding product in the company after subtracting the global sources costs calculated in the prices within the master problem. The whole figure, which is on the right side of the above-mentioned formula, defined the p-branch net profit within its production plan Xpj variant. Individual branches compare this figure to their profitability at the company level.

We know from the theory of duality that the above-mentioned formula duality subject is nothing more than the master problem optimal solution criteria. If those are not fulfilled, then the master problem at that stage is not optimal and can be refined. This will be achieved if the master problem is extended with the plan Xpj variant for which the *net* local profit is higher than the corresponding total profitability evaluation v_p where the Δ p

$$- \Delta_p = v_p - (c^p - uA_p)X_{pj} = v_p - L_p$$

defines the effect (expressed in the total company profit increase) achieved by the plan variant xpj being taken into consideration at the company level (i.e. while the corresponding master problem is being solved).

It means that every branch, knowing global sources optimal prices and taking its local conditions bp a bp into account, tries to make its plan better, i.e. to find the plan variant that would guarantee the maximum profit at estimated prices.

$$t_p(x^p) = (c^p - uA_p)x^p \rightarrow \max$$

As soon as the branches find these (more precise) plan variants, these are reported to the centre company. The company re-calculates the master plan in accordance with the obtained plan variant. The master plan is then extended using the plan variant xrj (r-branch, j-variant), in which the net local profit exceeds the corresponding total profitability evaluation, i.e.:

$$\Delta_R = \mathrm{MAX}\ \{L_P - V_P \mid P = 1, \ldots, K\} > 0$$

or

$$-\Delta_R = \mathrm{MIN}\ (V_P - L_P \mid P = 1, \ldots, K) < 0$$

The result of the extended master plan solution is adjusted production plans for individual branches, new global sources dual prices and new branches profitability total evaluation.

Then it all goes back to the branches and the interactive information exchange between the company management and the branches takes place until the optimal company plan is found, i.e. until it is not possible to rise the company total profit, i.e. until the following applies

$$\Delta_r = L_r - v_r = \max\ \{L_p - v_p \mid p = 1, \ldots, k\} \leq 0$$

or

$$-\Delta_r = v_r - L_r = \min\ \{v_p - L_p \mid p = 1, \ldots, k\} \geq 0$$

The above mentioned schedule of interconnected optimal company system) and branches (subsystem) plan puts into effect a fundamental optimal planning theory assumption; the plan, prices and economy activity simulation indicators must be interconnected and they must get a unique optimal solution to the production problem.

When constructing the production complex planning real scheme according to the decomposition method it must be emphasized that LPT with the block structure is a very simplified and idealized description of the real model. However, the block optimization very principles are vitally important for the production complex planning construction.

Conclusions

The Dantzing-Wolf decomposition algorithm used in two-stage problem optimization was originally designed for extensive linear programming optimization tasks. The initial extensive optimization task was divided into several smaller sub problems, which were solved independently, and in the end the optimal solution was obtained by all sub problems partial optimal solutions aggregation. Hardware and software technology development has put an end into this kind of application. The Dantzing-Wolf algorithm can still be used in a two-stage optimization. The algorithm in the article is described from the theoretical point of view but it is not difficult to solve a practical situation using one of optimization packets, e.g. LINDO, LINGO, SOLVER, etc. Quite interesting results will be obtained this way. If we optimize activities in companies individually, i.e. we construct and solve the problems for every company apart, we will get a certain profit mass[7]. But after the optimization made by the decomposition methods, the profit sum will be higher for all the

[7] special – purpose, criterial function was profit maximization

companies together with the parent company. However, the situation may occur that one of the company will show a lower profit that it could be if the company optimized its activity individually. Last but not least, it must be said that the Dantzing–Wolf algorithm does not always have a solution.

REFERENCES

1. Klieštik T. a kol. Modely hodnotenia efektívnosti podnikov v odbore cestnej dopravy pomocou analýzy obalu dát (DEA), projekt VEGA 1/3777/06
2. Dantzing G. B. Lineárne programovanie a jeho rozvoj, SVLT, Bratislava 1966
3. Churchmann C. W., Ackoff R.L , Arnoff E. L. Úvod do operačného výskumu, Alfa, Bratislava 1968
4. Mlynarovič V. a kol. Algoritmizácia modelov, ES VŠE, Bratislava 1989
5. Solver Suite: LINGO, LINDO, What´s Best, Lingo systems Inc., Chicago 1995
6. Bartošová V. Účtovníctvo - teória a prax s aplikáciou na dopravný podnik*,* Žilina: Žilinská univerzita v Žiline, 2001, 247 s.
7. Laščiak a kol. Optimálne programovanie, Alfa, Bratislava, 1990